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SYNTHESIS OF OPTIMAL FEEDBACK AND THE STABILIZATION OF SYSTEMS WITH A DELAY IN THE CONTROL†

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The linear problem of the optimal control of systems in which the input signals contain a time delay is considered. The method of realizing optimal feedback control that is proposed is based on a special procedure for correcting the current optimal programme controls, realized by an optimal controller using a dual linear programming method. The results are used to construct two types of stabilizer of systems with a delay in the control. © 1998 Elsevier Science Ltd. All rights reserved.

A new approach to the problem of synthesizing optimal systems when the optimized systems are described by ordinary differential equations has been described in [1–3]. Our aim here is to show how the basic constructions of [1–3] can be applied to control systems in which the mathematical models include a delay. The results on the synthesis of optimal systems can be used to design bounded stabilizing feedback. Unlike the approach to the stabilization of systems with a delay in the feedback channel used in [4], bounded stabilizing feedback achieved using a dual linear programming method.

1. STATEMENT OF THE PROBLEM

We consider a control system whose behaviour in the time interval $T = [0, t^*]$ is described by the equation

$$\dot{x}(t) = Ax(t) + bu(t) + b_1u(t-h) \tag{1.1}$$

where $x(t)$ is the n-vector of state of the system at time t, $u(t)$ is the value of the scalar control effect and $h > 0$ is the delay. We shall assume that system (1.1) is relatively controllable [5].

The class of accessible controls consists of piecewise-constant functions with quantization period $\nu = t^*/N = h/M$, N, M are natural numbers, and $u(t) = u_k$, $t \in [k\nu, (k+1)\nu[(k = 0, 1, \ldots, N-1)$, constrained by the bound

$$|u(t)| \leqslant L, \quad t \in T \tag{1.2}$$

We specify the terminal constraint as

$$Hx(t^*) = g \quad (H \in R^{m\times n}, \operatorname{rank} H = m < n) \tag{1.3}$$

An accessible control $u(t)$, $t \in T$ is said to be admissible if the corresponding path $x(t)$, $t \in T$ of system (1.1) with the initial condition $x(0) = x_0$, $u(t) = u_0(t)$, $t \in [-h, 0[$ satisfies terminal constraint (1.3).

On the set of admissible controls, we define the quality criterion

$$J(x) = c'x(t^*) \tag{1.4}$$

and optimal program control $u^0(t)$, $t \in T$: $J(u^0) = \max J(u)$.

We now embed problem (1.1)–(1.4) in the family of problems

$$c'x(t^*) \to \max, \quad \dot{x}(t) = Ax(t) + bu(t) + b_1u(t-h)$$
$$x(\tau) = z, \quad u(t) = \upsilon(t), \quad t \in [\tau - h, \tau[$$

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$$Hx(t^*) = g, \quad |u(t)| \leq L, \quad t \in T_\tau = [\tau, t^*] \tag{1.5}$$
$$\tau = k\nu, \quad k = 0, 1, \dots, N-1, \quad z \in R^n$$

where $\upsilon(t)$, $t \in [\tau - h, \tau[$ is a piecewise-continuous function, $|\upsilon(t)| \leq L$, $t \in [\tau - h, \tau[$.

The optimal program control of problem (1.5) is denoted by $u^0(t \mid \tau, z, \upsilon_\tau(\cdot))$, $t \in T_\tau$, ($\upsilon_\tau(\cdot) = (\upsilon(t), t \in [\tau - h, \tau[)$). Let Ω_τ be the set of all pairs $(z, \upsilon_\tau(\cdot))$ for which problem (1.5) has a solution.

The functional

$$u^0(\tau, z, \upsilon_\tau(\cdot)) = u^0(\tau \mid \tau, z, \upsilon_\tau(\cdot)), \quad (z, \upsilon_\tau(\cdot)) \in \Omega_\tau \tag{1.6}$$
$$\tau \in T^\nu = \{0, \nu, \dots, (N-1)\nu\}$$

will be called the optimal feedback control (the positional solution of problem (1.5)).

To construct functional (1.6) in explicit (formulaic) form, as the classical statement of the problem of optimal synthesis expects, is a difficult problem which has still not been solved even for ordinary systems (1.1), $h = 0$. The difficulties that the solution of this problem encounter, both in the theory of the maximum principle and in dynamic programming, are printed out in [2]. Since they are fundamental and there is no prospect of solving them in the foreseeable future, it seems reasonable at this stage to modify the actual formulation of the problem.

The newly formulated problem of optimal synthesis is as follows [1]. Assume that the optimal feedback (OFB) (1.6) has been constructed. We use it to close system (1.1) and consider the behaviour of the closed system under constantly operating perturbations. The introduction of perturbations is not accidental. A feature of classical OFB [6] is that although it is defined by a determinate system, it is intended for the operation of a system with unknown perturbations. If no perturbations were assumed in the introduction of the feedback, there would be no point in having OFB, because the optimal program control works perfectly well in ideal conditions.

We will therefore consider the behaviour of the system

$$\dot{x}(t) = Ax(t) + bu^0(t, x(t), u_t(\cdot)) + b_1 u_0(t-h) + w(t), \quad x(0) = x_0, \quad t \in [0, h[$$
$$\dot{x}(t) = Ax(t) + bu^0(t, x(t), u_t(\cdot)) + b_1 u^0(t-h, x(t-h), u_{t-h}(\cdot)) + w(t), \quad t \in [h, t^*[\tag{1.7}$$

where $u_t(\cdot) = (u(s), s \in [t-h, t[)$, $u(s) = u^0(s, x(s), u_s(\cdot))$ is the control used at the previous instant s, $u(s) = u_0(s)$, $s \in [-h, 0[$, $w(t)$, $0 \leq t \leq t^0$, $w(t) \equiv 0$, $t > t^0$, is the unknown piecewise-continuous n-vector function of the perturbations and $t^0 = N_1\nu$, $N_1 < N - M$ is the time at which the operation of perturbations ceases.

We will denote by $w^*(t)$, $t \in T^0 = [0, t^0]$, a perturbation realized in some fixed control process. By (1.7), it will correspond to the path $x^*(t)$, $t \in T$, of the system to be optimized (assuming for simplicity that the control (1.7) has a solution). It is clear from (1.7) that a control $u^*(t) = u^0(t, x^*(t), u^*_t(\cdot))$, $t \in T$ will be received at the input of the system, so that in any specific control process we need to know not the entire OFB, but its values along an isolated curve $(x^*(t), u^*_t(\cdot))$, $t \in T$. Moreover, we do not need to know the value $u^*(\tau) = u^0(\tau, x^*(\tau), u^*_\tau(\cdot))$ before the time τ; we need only know how to compute it when the system is in the current position $(\tau, x^*(\tau), u^*_\tau(\cdot))$. This is clearly very much simpler than the initial problem of constructing the functional (1.6). We shall show below how to solve this problem using modern computational techniques.

We begin by introducing a new concept. Any device which is capable of computing the realization $u^*(t)$, $t \in T$, of OFB (1.6) in each specific process in real time will be called an optimal regulator. Thus, we have reduced the problem of realizing the OFB to describing the algorithm of operation of an optimal controller.

2. THE OPERATING ALGORITHM OF AN OPTIMAL CONTROLLER

Suppose that the controller has been constructed and has worked at times $0, \nu, \dots, (k-1)\nu$, and that system (1.7) was in state $x^*(k\nu)$ at time $\tau = k\nu$. To work out the control $u^*(t)$, $t \in [k\nu, (k+1)\nu[$ at time $\tau = k\nu$, it uses the optimal program control of the problem

$$
\begin{aligned}
&c'x(t^*) \to \max, \quad \dot{x}(t) = Ax(t) + bu(t) + b_1 u(t-h) \\
&x(k\nu) = x^*(k\nu), \quad u(t) = u^*(t), \quad t \in [k\nu - h, k\nu[\\
&Hx(t^*) = g, \quad |u(t)| \leqslant L, \quad t \in T_{k\nu}
\end{aligned}
\tag{2.1}
$$

In functional form, problem (2.1) has the form

$$
\begin{aligned}
&c' \int_\tau^{t^*} F(t^* - t)(bu(t) + b_1 u(t-h))\,dt \to \max \\
&HF(t^* - \tau)\,x^*(\tau) + H \int_\tau^{t^*} F(t^* - t)(bu(t) + b_1 u(t-h))\,dt = g \\
&u(t) = u^*(t), \quad t \in [\tau - h, \tau[; \quad |u(t)| \leqslant L, \quad t \in T_\tau
\end{aligned}
$$

where $F(t)$, $t \geqslant 0$ is the fundamental matrix of solutions of the homogeneous system $\dot{x} = Ax$.

This is equivalent to the following linear programming problem

$$
\begin{aligned}
&\sum_{i=1}^{K-M} c' \int_{(k+i-1)\nu}^{(k+i)\nu} (F(t^* - t)\,b + F(t^* - t - h)\,b_1)\,dt\,\xi_i + \sum_{K-M+1}^{K} c' \int_{(k+i-1)\nu}^{(k+i)\nu} F(t^* - t)\,b\,dt\,\xi_i \to \max \\
&\sum_{i=1}^{K-M} H \int_{(k+i-1)\nu}^{(k+i)\nu} (F(t^* - t)\,b + F(t^* - t - h)\,b_1)\,dt\,\xi_i + \sum_{K-M+1}^{K} H \int_{(k+i-1)\nu}^{(k+i)\nu} F(t^* - t)\,b\,dt\,\xi_i = \\
&= g - HF(t^* - k\nu)\,x^*(\tau) - H \int_{k\nu}^{(k+1)\nu} F(t^* - t)\,b_1 u^*(t-h)\,dt \\
&|\xi_i| \leqslant L, \quad i = 1, 2, \ldots, K, \quad K = N - k
\end{aligned}
\tag{2.2}
$$

Problem (2.2) contains m basic constraints and K variables. Let $\xi^0(\tau) = (\xi_i^0(\tau),\ i = 1, 2, \ldots, K)$ denote the optimal plan of problem (2.2) and $K^0(\tau)$ the optimal support of problem (2.2) [7]. Then the optimal program control in problem (2.1) will be $u^0(t) = u^0_{k+i-1} = \xi_i^0(\tau)$, $t \in [(k + i - 1)\nu, (k + i)\nu[$, $i = 1, 2, \ldots, K$.

At time $\tau = (k + 1)\nu$ the controller has to solve problem (2.2) for $K := K - 1$ and the new state $x^*(\tau)$, that is, the number of variables in problem (2.2) is one less and the vector of the right-hand sides changes (the smaller ν is, the less the vector changes). Thus there is no need to solve the linear programming problem completely at time $(k + 1)\nu$. If the optimal support $K^0(k\nu)$ is used as the initial approximation for time $(k + 1)\nu$, then problem (2.2) can be solved for time $(k + 1)\nu$ much more quickly by the dual linear programming method [7] than when there is no information about the support $K^0(k\nu)$. Thus, having constructed the optimal program control for time $\tau = 0$ before the start of the process, at each instant $k\nu$ $(k = 1, 2, \ldots, N_1)$ the controller corrects the existing optimal control for time $(k - 1)\nu$ and in the interval $[k\nu, (k + 1)\nu[$ sends the value $u^*(t) = \xi_i^0(k\nu)$, $t \in [k\nu, k + 1)\nu[$ to the input of system (1.7).

If the time taken to compute the control $u^*(t)$, $t \in [\tau, \tau + \nu[$, is no greater than ν, we say that the controller computes the realization of the OFB in real time.

Once the perturbations have ceased, the controller uses the optimal program control for time $N_1\nu$ and sends the values $u^*(t) = \xi_i^0(N_1\nu)$, $t \in [(N_1 + i - 1)\nu, (N_1 + i)\nu[$ $(i = 1, 2, \ldots, N - N_1)$ to the input of system (1.7).

Example. Consider the problem of the displacement of a system of two point masses connected by an elastic spring (Fig. 1). The equations of motion of the system are

$$
\begin{aligned}
&\dot{x}_1(t) = x_3(t), \quad \dot{x}_2(t) = x_4(t) \\
&\dot{x}_3(t) = -\frac{k}{m}(x_1(t) - x_2(t)) + u(t-h), \quad \dot{x}_4(t) = \frac{k}{M}(x_1(t) - x_2(t))
\end{aligned}
$$

where x_1, x_2 are deviations from the state of rest of the first and second objects, x_3, x_4 are their velocities, m and M are the masses of the objects and k is the coefficient of elasticity of the spring.

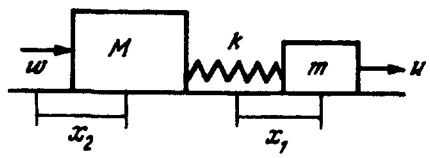

Fig. 1.

Using control effects of minimum intensity it is required to move this system to a new position in a fixed time t^* and stay there. Having chosen the fixed initial data, we obtain the following optimal control problem

$$
\begin{aligned}
&\rho \to \min \\
&\dot{x}_1(t) = x_3(t), \quad \dot{x}_2(t) = x_4(t) \\
&\dot{x}_3(t) = -(x_1(t) - x_2(t)) + u(t-h), \quad \dot{x}_4(t) = 2(x_1(t) - x_2(t)) \\
&x_1(0) = 0{,}5, \quad x_2(0) = 0{,}4, \quad x_3(0) = 0{,}2, \quad x_4(0) = -0{,}1 \\
&x_1(8) = 1, \quad x_2(8) = 1, \quad x_3(8) = 0, \quad x_4(8) = 0 \\
&u(t) \equiv 0, \quad t \in [-h, 0[, \quad |u(t)| \leqslant \rho, \quad t \in [0, 8-h]
\end{aligned}
\tag{2.3}
$$

The results of the programmed solution of problem (2.3) for different delays h are shown in Figs 2(a) and 3(a), which depict the phase trajectories of the objects, and in Fig. 4(a), in which the corresponding programmed controls are represented by the solid lines for $h = 0$ and by the dashed lines for $h = 2$.

Suppose that a perturbation $w(t)$, $t \in [0, 6]$ which is unknown to the controller acts on the second object (Fig. 1), and the behaviour of the system is described by the equations

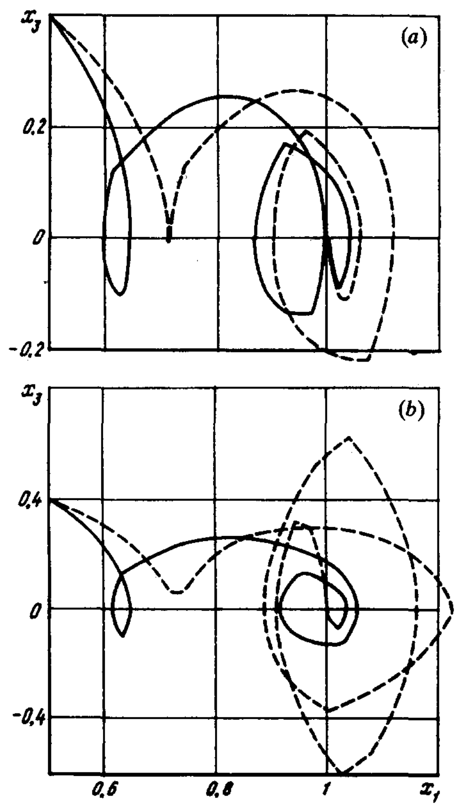

Fig. 2.

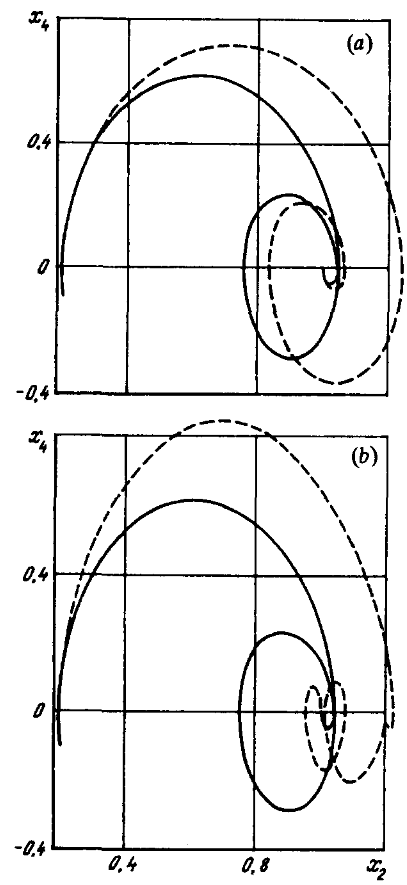

Fig. 3.

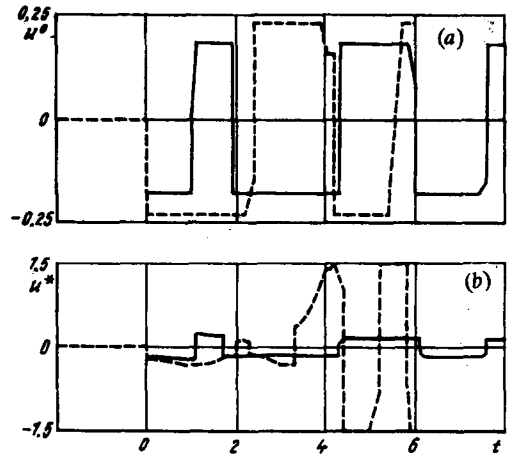

Fig. 4.

$$\dot{x}_1(t)=x_3(t),\quad \dot{x}_2(t)=x_4(t)$$

$$\dot{x}_3(t)=-(x_1(t)-x_2(t))+u(t-h),\ \dot{x}_4(t)=2(x_1(t)-x_2(t))+w(t)$$

The results of the operation of the controller for $w(t) = 0.4\sin 3t, t \in [0, 6]$ are shown in Fig 2(b) and 3(b) (the phase trajectories of the objects) and in Fig. 4(b) (the controls $u^*(t), t \in [0, 8[$) worked out by the controller). The same notation is used as in Figs 2(a), 3(a) and 4(a).

3. THE USE OF OPTIMAL FEEDBACK TO STABILIZE DYNAMIC SYSTEMS WITH A DELAY IN THE CONTROL

Consider a dynamical system, the behaviour of which together with the applied control can be described for $t \geqslant 0$ by the equation

$$\dot{x}(t)=Ax(t)+bu(t)+b_1u(t-h),\quad x(0)=x_0 \tag{3.1}$$
$$u(t)=u_0(t),\quad t\in[-h,0[$$

Assuming that the intrinsic dynamics of the system (when $u(t) \equiv 0, t \geqslant 0$) is unstable, we have the problem of stabilizing it by bounded controls.

Let G be some neighbourhood of the state of equilibrium $x = 0$ of system (3.1). The functional $u(x, \upsilon(\cdot)), x \in G, \upsilon(\cdot) \in V = \{\upsilon(\cdot) = (\upsilon(t), t \in [-h, 0[), \mid \upsilon(t) \mid \leqslant L, t \in [-h, 0[\}$ will be called a bounded stabilizing feedback (BSFB) if (1) $u(0, 0) = 0$, (2) $\mid u(x, \upsilon(\cdot)) \mid \leqslant L, x \in G, \upsilon(\cdot) \in V$, (3) the closed system

$$\begin{aligned}&\dot{x}(t)=Ax(t)+bu(x(t),v_t(\cdot))+b_1u_0(t-h)\\&x(0)=x_0,\quad t\in[0,h[\\&\dot{x}(t)=Ax(t)+bu(x(t),v_t(\cdot))+b_1u(x(t-h),v_{t-h}(\cdot)),\quad t\geqslant h\end{aligned} \tag{3.2}$$

where $u^*(t), t \geqslant 0$ is the control applied to the system at time t, $u^*(t) = u^0(t), t \in [-h, 0[, \upsilon_t(\cdot) = (\upsilon(s) = u^*(t + s), s \in [-h, 0[)$, for all $x_0 \in G, u^0(\cdot) \in V$ has a unique continuous solution $x(t), t \geqslant 0$, (4) the zero solution $x(t) \equiv 0, t \geqslant 0$ of system (3.2) is asymptotically stable in $G \times V$.

Unlike the problems of optimal control studied in previous sections, the stabilization problem (the construction of the BSFB) is defined over the entire time axis $t \geqslant 0$. However, a shown for ordinary dynamical systems in [3], it can be solved by means of optimal control with a finite horizon using the principle of "sliding" control.

We will show that this approach is also successful in the case of system (3.1) with a delaying control. To do so, we will analyse how the BSFB is used in a specific stabilization process. Let $x^*(t), t \geqslant 0$ denote the path of the system being stabilized, and let $\upsilon_t^*(\cdot)$ denote the set of values of the control $u^*(t + s)$, $s \in [-h, 0[$ sent to the input of system (3.2) in the interval $[t-h, t[$. It is clear from the above definition of the BSFB that for a specific stabilization process, the values of the functional $u(x, \upsilon(\cdot)), x \in G, \upsilon(\cdot) \in V$ are needed only for specific realizations $x^*(t), u^*(t), t \geqslant 0$. Moreover, we do not need to know the value of $u^*(\tau) = u^0(x^*(\tau), \upsilon^*_\tau(\cdot))$ beforehand, but merely how to compute it at time τ. We call the function $u^*(t) = u^*(k\nu), t \in [k\nu), (k + 1)\nu[\ (k = 0, 1, \ldots)$ a discrete realization of the BSFB, and a device which, in each specific process, for chosen ν is capable of computing the values $u^*(k\nu)$ $(k = 0, 1, \ldots)$ in real time will be called a (discrete) stabilizer.

Thus, the stabilization of dynamical systems with a control delay has involved the construction of an algorithm to operate the stabilizer. The algorithm is based on the operating algorithm of an optimal controller for a special optimal control. We will consider two types of stabilizer, defined by the quality criterion of an auxiliary optimal control.

The first type of stabilizer. We will choose the parameter of the method $\Theta = N\nu$ and introduce the auxiliary (accompanying) optimal control problem [8]

$$\begin{aligned} &\rho(z, \upsilon(\cdot)) = \min \rho \\ &\dot{x}(t) = Ax(t) + bu(t) + b_1 u(t-h), \quad x(0) = z \\ &u(t) = \upsilon(t), \quad t \in [-h, 0[, \quad x(\Theta) = 0 \\ &|u(t)| \leqslant \rho, \quad t \in [0, \Theta - h[, \quad u(t) \equiv 0, \quad t \in [\Theta - h, \Theta[\end{aligned} \tag{3.3}$$

We call the functional $u^0(t \mid z, \upsilon(\cdot)), t \in [0, \Theta - h[$ an optimal control of problem (3.3) if the path $x^0(t \mid z, \upsilon(\cdot)), t \in [0, \Theta]$ that it generates satisfies the constraint $x^0(\Theta \mid z, \upsilon(\cdot)) = 0$ and the quality criterion has the minimum value.

The functional $u^0(x(t), \upsilon_t(\cdot)) = u^0(0 \mid x(t), \upsilon_t(\cdot)), x(t) \in R^n, \upsilon_t(\cdot) \in V, t \geqslant 0$ is a BSFB.

Indeed, it follows from the conditions of the auxiliary problem that: (1) $u^0(0, 0) = 0$; (2) for each bounded set $G \subset R^n$ there is a number $L > 0$ for which $\rho(x, \upsilon(\cdot)) \leqslant L$ for all $x \in G, \upsilon(\cdot) \in V$, and thus also $|u^0(x, \upsilon(\cdot))| \leqslant L$. The closed system (3.2) $(u(x(t), \upsilon_t(\cdot)) = u^0(x(t), \upsilon(\cdot)))$ is asymptotically stable. In order to show this, we will consider an arbitrary time $\tau = l\nu$, state $x^*(\tau) = x^*(\tau \mid x_0, u_0(\cdot))$ and computed control $u^*_\tau(\cdot) = (u^*(t), t \in [\tau - h, \tau[)$ corresponding to an arbitrary initial state $x(0) = x_0 \in G$ and $u_0(\cdot) \in V$. For the position $x^*(\tau), u^*_\tau(\cdot)$ the quality criterion of problem (3.3) takes the value $\rho(x^*(\tau), u^*_\tau(\cdot))$. It is computed by solving problem (3.3) with $z = x^*(\tau), \upsilon(t) = u^*(\tau + 1), t \in [-h, 0[$. Clearly, for $x^*(\tau) = 0, u^*(\tau + t) = 0, t \in [-h, 0[$, we have the equation $\rho(x^*(\tau), u^*_\tau(\cdot)) = 0$. But if $x^*(\tau) \neq 0, u^*(\tau + t) \neq 0, t \in [-h, 0[$ then $\rho(x^*(\tau), u^*_\tau(\cdot)) > 0$.

In functional form problem (3.3) has the form

$$\begin{aligned} &\rho \to \min \\ &F(N\nu)x^*(\tau) + \int_0^{M\nu} F(N\nu - t) b_1 u^*(\tau - h + t)dt + \\ &+ \sum_{i=0}^{K-1} \int_{i\nu}^{(i+1)\nu} (F(N\nu - t)b + F(N\nu - t - h)b_1)dt\, u_i = 0 \\ &|u_i| \leqslant \rho, \quad i = 0, 1, \ldots, K-1, \quad K = N - M \end{aligned} \tag{3.4}$$

Introducing the new variables $\xi_0 = 1/\rho, \xi_i = u_{i-1}/\rho$ $(i = 1, 2, \ldots, K)$, we obtain the equivalent linear programming problem with $K + 1$ variables and n basic constraints

$$\begin{aligned} &\xi_0 \to \max \\ &(F(N\nu)x^*(\tau) + \int_0^{M\nu} F(N\nu - t) b_1 u^*(\tau - h + t)dt)\xi_0 + \\ &+ \sum_{i=1}^{K} \int_{(i-1)\nu}^{i\nu} (F(N\nu - t)b + F(N\nu - t - h)b_1)dt\, \xi_i = 0 \\ &\xi_0 \geqslant 0, \quad |\xi_i| \leqslant 1, \quad i = 1, 2, \ldots, K \end{aligned} \tag{3.5}$$

We will denote by $\xi^0(\tau) = (\xi^0_i(\tau), i = 0, 1, \ldots, K)$ the optimal plan of problem (3.5) and by $K^0(\tau)$ the optimal support of problem (3.5). Then the optimal control in problem (3.4) will be $u^0(\tau) = (u^0_i(\tau) = \xi^0_{i+1}(\tau)\xi^0_0(\tau), i = 0, 1, \ldots, K-1$.

At time $\tau + \nu$ the closed system (3.2) is in the state

$$x^*(\tau+\nu)=F(\nu)x^*(\tau)+\int_0^\nu F(\nu-t)bdtu_1^0(\tau)+\int_0^\nu F(\nu-t)b_1dtu^*(\tau-h)$$

We will show that the value of the quality criterion of problem (3.3) for the new position $z = x^*(\tau + \nu)$, $\upsilon(t) = u^*(\tau + \nu + t)$, $t \in [-h, 0[$ satisfies the inequality $\rho(x^*(\tau + \nu), u^*_{\tau+\nu}(\cdot)) \leqslant \rho(x^*(\tau), u^*_\tau(\cdot))$. For, under the effect of the control $u_i(\tau + \nu) = u^0_{i+1}(\tau)$ $(i = 0, 1, \ldots, K-2)$, $u_i(\tau + \nu) = 0$ $(i = K-1, K, \ldots, N-1)$, system (3.2) moves from the initial position $x^*(\tau + \nu)$, $u_{\tau+\nu}(\cdot)$ to the origin of coordinates and is held there, and the inequality $|u_i(\tau + \nu)| \leqslant \rho(x^*(\tau), u^*_\tau(\cdot))$ $(i = 0, 1, \ldots, N-1)$ holds for the control $u_i(\tau + \nu)$ $(i = 0, 1, \ldots, N-1)$. Hence, the inequality $\rho(x^*(\tau + \nu), u^*_{\tau+\nu}(\cdot)) = \rho(x^*(\tau), u^*_\tau(\cdot))$ will also hold on the optimal control $u^0_i(\tau + \nu)$ $(i = 0, 1, \ldots, N-1)$. It remains to show that the equation $\rho(x^*(\tau + \nu), u^*_{\tau+\nu}(\cdot)) = \rho(x^*(\tau), u^*_\tau(\cdot))$ can hold for not more than K steps.

If the equations

$$\rho(x^*(\tau),u_\tau^*(\cdot))=\rho(x^*(\tau+q\nu),u^*_{\tau+q\nu}(\cdot)),\quad q=1,2,\ldots,K-1 \tag{3.6}$$

are satisfied during stabilization, then the control $u_i(\tau + q\nu) = u^0_{i+q}(\tau)$ $(i = 0, 1, \ldots, K-q-1)$, $u_i(\tau + q\nu) = 0$ $(i = K-q, K-q+1, \ldots, N-1)$ will be an optimal program control in problem (3.5) for the position $z = x^*(\tau + q\nu)$, $\upsilon(t) = u^*(\tau + q\nu + t)$, $t \in [-h, 0[$ $(q = 1, 2, \ldots, K-1)$. If Eq. (3.6) is not violated for $q = K$, then the optimal program control for position $z = x^*(\tau + K\nu)$, $\upsilon(t) = u^*(\tau + K\nu + t)$, $t \in [-h, 0[$ will be $u_i(\tau + K\nu) = 0$ $(i = 0, 1, \ldots, N-1)$, which gives $\rho(x^*(\tau + K\nu), u^*_{\tau+K\nu}(\cdot)) = 0$. Thus, it is guaranteed that

$$\rho(x^*(\tau),u_\tau^*(\cdot))>\rho(x^*(\tau+K\nu),u^*_{\tau+K\nu}(\cdot)) \tag{3.7}$$

Using inequality (3.7), and arguments typical of the method of Lyapunov functions [9–12], it can be shown that $\rho(x^*(\tau + k\nu), u^*_{k\nu}(\cdot)) \to 0$, $\| x^*(k\nu) \| \to 0$, $k \to \infty$. Hence it follows that $\| x^*(t) \| \to 0$, $t \to \infty$, as it was required to prove.

We will now describe the operating algorithm of the stabilizer.

Suppose that the stabilizer has worked at times $0, \nu, \ldots, l\nu$. To work out the control $u^*(\tau)$ at time $\tau = l\nu$ it uses the values of the optimal program control $u^0(0\,|\,x^*(\tau), \upsilon_\tau(\cdot))$ of problem (3.3) for $z = x^*(\tau)$, $\upsilon(t) = u^*(\tau + t)$, $t \in [-h, 0[$. This is equivalent to the linear programming problem (3.5). To work out the control $u^*(\tau + \nu)$, the stabilizer needs to know the solution of problem (3.3) for the initial position $z = x^*(\tau + \nu)$, $\upsilon(t) = u^*(\tau + \nu + t)$, $t \in [-h, 0[$. This is equivalent to the following linear programming problem

$$\begin{aligned}
&\xi_0 \to \max\\
&(F(N\nu)x^*(\tau+\nu)+\int_0^{M\nu} F(N\nu-t)b_1u^*(\tau+\nu-h+t)dt)\xi_0+\\
&+\sum_{i=1}^{K}\int_{(i-1)\nu}^{i\nu}(F(N\nu-t)b+F(N\nu-t-h)b_1)dt\xi_i=0 \qquad (3.8)\\
&\xi_0\geqslant 0,\quad |\xi_i|\leqslant 1,\quad i=1,2,\ldots,K
\end{aligned}$$

The new problem differs from (3.5) only in the vector of the conditions for variable ξ_0, and the smaller the value of ν the less that difference is. As in the calculation of the realization of optimal feedback, problem (3.8) can be solved by the dual linear programming method using the optimal support of problem (3.5) $K^0(\tau)$. For small ν, the solution of the new problem is constructed in a small number of iterations. For large ν, the number of iterations of the dual method can be reduced by dividing the interval $[\tau, \tau + \nu]$ into enough small intervals that, by solving problems of type (3.5) by the dual method in those intervals, a small number of iterations is needed to compute the optimal control $u^0(t\,|\,x^*(\tau + \nu), \upsilon_{\tau+\nu}(\cdot))$, $t = \in [0, \Theta - h[$ at time $\tau + \nu$.

The method is illustrated by the problem of damping a two-mass oscillatory system (Fig. 5). Here m and M denote the masses, x_1, x_2 their coordinates, c_1, c_2 the coefficients of elasticity of the springs and u the damping action. The mathematical model of the system has the form

$$\dot{x}_1(t)=x_3(t),\quad \dot{x}_2(t)=x_4(t),\quad \dot{x}_3(t)=(-c_1x_1(t)+c_1x_2(t)+u(t-h))/m \tag{3.9}$$

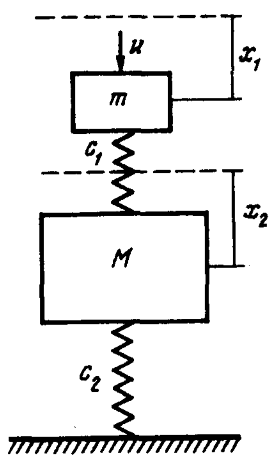

Fig. 5.

$$\dot{x}_4(t) = (c_1 x_1(t) - (c_1 + c_2) x_2(t)) / M$$

The following parameter values are assigned: $m = 1, M = 10, c_1 = 1, c_2 = 9.2$. We take the initial state as $x_0 = (0.5; 0.4; 0.2; -0.1)$. In this case auxiliary problem (3.3) has the form

$$\begin{aligned}
&\rho \to \min, \quad \dot{x}_1(t) = x_3(t), \quad \dot{x}_2(t) = x_4(t)\\
&\dot{x}_3(t) = -x_1(t) + x_2(t) + u(t-h), \quad \dot{x}_4(t) = 0{,}1x_1(t) - 1{,}02x_2(t)\\
&x(0) = z, \quad u(t) = 0, \quad t \in [-h, 0[, \quad x(\Theta) = 0\\
&|u(t)| \leqslant \rho, \quad t \in [0, \Theta - h[
\end{aligned}$$

The parameters of the problem were given the values $\Theta = 4$, $\nu = 0.1$. The system was damped with different delays. Figure 6 shows the realized states $x^*_1(\tau)$, $\tau \geqslant 0$ corresponding to each value of h, the realized controls $u^*(\tau)$, $\tau \geqslant -h$, and the behaviour of the Lyapunov function $\rho(x^*(\tau), u^*_\tau(\cdot))$ for each of the processes (the solid lines for $h = 0$ and the dashed lines for $h = 1$).

The second type of stabilizer. The operating algorithm of this stabilizer is based on the solution of an auxiliary optimal control problem [8] of the type

$$\begin{aligned}
&\mu(z, \upsilon(\cdot)) = \min \int_0^{\Theta - h} |u(t)|\, dt\\
&\dot{x}(t) = Ax(t) + bu(t) + b_1 u(t-h) \quad x(0) = z\\
&u(t) = \upsilon(t), \quad t \in [-h, 0[, \quad x(\Theta) = 0\\
&|u(t)| \leqslant L, \quad t \in [0, \Theta - h[, \quad u(t) = 0, \quad t \in [\Theta - h, \Theta[
\end{aligned} \tag{3.10}$$

As in the case of the first type of stabilizer, $u^0(t \mid z, \upsilon(\cdot))$, $t \in [0, \Theta - h[$ denotes the optimal program control of problem (3.10). Let G be the set of all $z \in R^n$ for which problem (3.10) has a solution.
The functional $u^0(x(t), \upsilon_t(\cdot)) = u^0(0 \mid x(t), \upsilon_t(\cdot)), x(t) \in G, \upsilon_t(\cdot) \in V, t \geqslant 0$, is the BSFB.

For the proof, we take the Lyapunov function as the optimal value of the quality criterion in problem (3.10). Clearly $\mu(0, 0) = 0, \mu(x, \upsilon(\cdot)) > 0, x \neq 0, \upsilon(\cdot) \not\equiv 0$.
Suppose at time $\tau = l\nu$ in problem (3.10), for $z = x^*(\tau)$, $\upsilon(t) = u^*(\tau + t)$, $t \in [-h, 0[$, that we have constructed the optimal program control $u^0(\tau) = (u_i^0(\tau), i = 0, 1, \ldots, K-1)$, in which the quality criterion has the value

$$\mu(x^*(\tau), u^*_\tau(\cdot)) = \sum_{i=0}^{K-1} |u_i^0(\tau)| > 0$$

At time $\tau + \nu$ the system is in the state

$$x^*(\tau+\nu)=F(\nu)x^*(\tau)+\int_0^{\nu}F(\nu-t)b\,dt\,u_1^0(\tau)+\int_0^{\nu}F(\nu-t)b_1dt\,u^*(\tau-h)$$

In problem (3.10) for $z = x^*(\tau+\nu)$, $\upsilon(t) = u^*(\tau+\nu+t)$, $t \in [-h, 0[$, the control

$$u_i(\tau+\nu)=u_{i+1}^0(\tau),\quad i=0,1,\ldots,K-1;\quad u_i(\tau+\nu)=0,\quad i=K,K+1,\ldots,N \tag{3.11}$$

will be admissible, and

$$\sum_{i=1}^{K}|u_i(\tau+\nu)|=\mu(x^*(\tau),u_\tau^*(\cdot))-|u_0^0(\tau)|$$

Clearly, if $u_0^0(\tau) \neq 0$, then $\mu(x^*(\tau+\nu), u^*_{\tau+\nu}(\cdot)) < \mu(x^*(\tau), u^*_\tau(\cdot))$. If $u_1^0(\tau) = 0$ and $\mu(x^*(\tau+\nu), u^*_{\tau+\nu}(\cdot)) = \mu(x^*(\tau), u^*_\tau(\cdot))$, then in problem (3.10) for $z = x^*(\tau+\nu)$, $\upsilon(t) = u^*(\tau+\nu+t)$, $t \in [-h, 0[$, the control (3.11) is the optimal program control. By virtue of the inequality $\mu(x^*(\tau), u^*_\tau(\cdot)) > 0$, there is a number $s < K$ for which $u_s^0(\tau) \neq 0$, ensuring that the inequality $\mu(x^*(\tau+(s+1)\nu), u^*_{\tau+(s+1)\nu}(\cdot)) < \mu(x^*(\tau), u^*_\tau(\cdot))$ is satisfied.

Thus the Lyapunov function $\mu(x^*(\tau), u^*_\tau(\cdot))$ decreases monotonically along the sequence $x^*(kK\nu)$, $u^*_{kK\nu}(\cdot)$ ($k = 0, 1, \ldots$). By the usual argument in the method of Lyapunov functions [10–13], it can be shown that the limit must be zero: $\lim \mu(x^*(kK\nu), u^*_{kK\nu}(\cdot)) = 0$, $k \to \infty$. Hence it follows that $\|x^*(kK\nu)\| \to 0$, $k \to \infty$ and $\|x^*(t)\| \to 0$, $t \to \infty$, which it was required to prove.

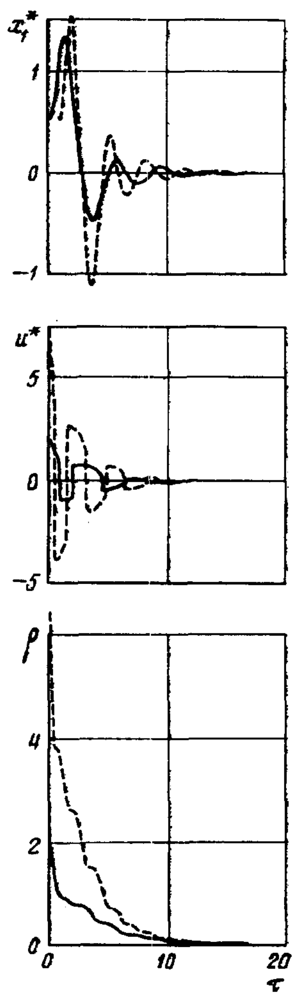

Fig. 6.

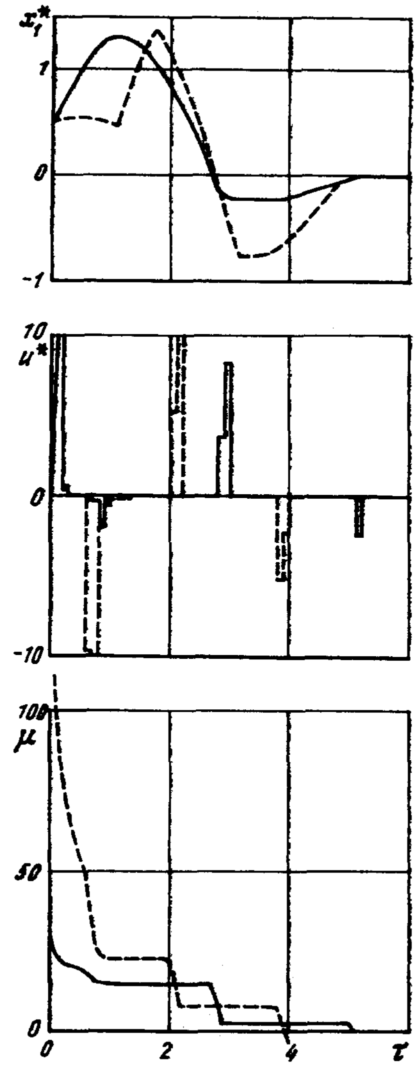

Fig. 7.

Since $\mu(x^*(\tau+\nu), u^*_\tau(\cdot)) \leq \mu(x^*(\tau+\nu), u^*_\nu(\cdot)) - |u^*(\tau)|$, it can be shown that to realize stabilizing feedback $u^*(k\nu)$ $(k = 0, 1, \dots)$ the following property must hold

$$\sum_{k=0}^{\infty} |u^*(k\nu)| \leq \sum_{k=0}^{K-1} |u^0(k\nu)|$$

where $u^0(k\nu)$ $(k = 0, 1, \dots, K-1)$ is the optimal program control in problem (3.10) for the initial position $z = x_0$, $\upsilon(t) = u_0(t)$, $t \in [-h, 0[$. Thus, even before the start of the stabilization process, we can compute the maximum value of the requisite control resources.

We will now describe the operating algorithm of the stabilizer.

The optimal control problem (3.10) is equivalent to the linear programming problem

$$\begin{aligned}
&\sum_{i=1}^{K} (\upsilon_i + w_i) \to \min\\
&\sum_{i=1}^{K} \int_{(i-1)\nu}^{i\nu} (F(N\nu - t)b + F(N\nu - t - h)b_1)\,dt(\upsilon_i - w_i) =\\
&= -F(N\nu)z - \int_0^{M\nu} F(N\nu - t)b_1\upsilon(\tau - h + t)\,dt \qquad (3.12)\\
&\upsilon_i + w_i \leq L, \quad \upsilon_i \geq 0, \quad w_i \geq 0, \quad i = 1, 2, \dots, K
\end{aligned}$$

At each time $\tau = k\nu$, the stabilizer uses the dual method to construct the solution $(\upsilon_i^0, w_i^0, i = 1, 2, \dots, K)$ of problem (3.12) for $z = x^*(\tau)$, $\upsilon(t) = u^*(\tau + t)$, $t \in [-h, 0[$, taking the initial support as the optimal support of problem (3.12) for $z = x^*(\tau - \nu)$, $\upsilon(t) = u^*(\tau - \nu + t)$, $t \in [-h, 0[$. The control $u^*(t) = \upsilon_1^0 - w_1^0$ is sent to the input of the system in the interval $[\tau, \tau + \nu[$.

As an illustration we will consider the damping of system (3.9) using the second type of stabilizer. In this case the auxiliary problem has the form

$$\begin{aligned}
&\int_0^{\Theta-h} |u(t)| \to \min, \quad \dot{x}_1(t) = x_3(t), \quad \dot{x}_2(t) = x_4(t)\\
&\dot{x}_3(t) = -x_1(t) + x_2(t) + u(t-h), \quad \dot{x}_4(t) = 0{,}1x_1(t) - 1{,}02x_2(t)\\
&x(0) = z, \quad u(t) = 0, \quad t \in [-h, 0[, \quad x(\Theta) = 0\\
&|u(t)| \leq 10, \quad t \in [0, \Theta - h[
\end{aligned}$$

The parameters of the problem are given the same values as before.

Figure 7 shows the results of damping with different delays the realized states $x_1^*(\tau)$, $\tau \geq 0$, worked out by the stabilizer of control $u^*(\tau)$, $\tau \geq -h$, and the values of the Lyapunov function $\mu(x^*(\tau), u^*_\tau(\cdot))$. The same notation is used as in Fig. 6. For each process we have calculated the values of the sum of the moduli of the realized controls

$$s = \Sigma_{k=0}^{89} |u^*(0.1k)|\text{: (1) } h = 0, s = 27.45\text{; (2) } h = 1, s = 64.69.$$

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